

Decomposing 8-regular graphs into paths of length 4

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Abstract

A T -decomposition of a graph G is a set of edge-disjoint copies of T in G that cover the edge set of G . Graham and Häggkvist (1989) conjectured that any 2ℓ -regular graph G admits a T -decomposition if T is a tree with ℓ edges. Kouider and Lonc (1999) conjectured that, in the special case where T is the path with ℓ edges, G admits a T -decomposition \mathcal{D} where every vertex of G is the end-vertex of exactly two paths of \mathcal{D} , and proved that this statement holds when G has girth at least $(\ell + 3)/2$. In this paper we verify Kouider and Lonc's Conjecture for paths of length 4.

Keywords: Decomposition, regular graph, path

1 Introduction

A decomposition of a graph G is a set \mathcal{D} of edge-disjoint subgraphs of G that cover the edge set of G . Given a graph H , we say that \mathcal{D} is an H -decomposition of G if every element of \mathcal{D} is isomorphic to H . Ringel [12] conjectured that the complete graph $K_{2\ell+1}$ admits a T -decomposition for any tree T with ℓ edges. Ringel's Conjecture is commonly confused with the *Graceful Tree Conjecture* that says that any tree T on n vertices admits a labeling $f: V(T) \rightarrow \{0, \dots, n-1\}$ such that $\{1, \dots, n-1\} \subseteq \{|f(x) - f(y)|: xy \in E(T)\}$. Since the Graceful Tree Conjecture implies Ringel's Conjecture [13], Ringel's Conjecture holds for many classes of trees such as stars, paths, bistars, caterpillars, and lobsters (see [3, 6]). Häggkvist [7] generalized Ringel's Conjecture for regular graphs as follows.

Conjecture 1.1 (Graham–Häggkvist, 1989). *Let T be a tree with ℓ edges. If G is a 2ℓ -regular graph, then G admits a T -decomposition*

Häggkvist [7] also proved Conjecture 1.1 when G has girth at least the diameter of T . For more results on decompositions of regular graphs into trees, see [4, 5, 8, 9]. For the case where $T = P_\ell$ is the path with ℓ edges (note that this notation is not standard), Kouider and Lonc [10] improved Häggkvist's result proving that *if G is a 2ℓ -regular graph with girth $g \geq (\ell + 3)/2$, then G admits a balanced P_ℓ -decomposition \mathcal{D}* , that is a path decomposition \mathcal{D} where each vertex is the end-vertex of exactly two paths of \mathcal{D} . These authors also stated the following strengthening of Conjecture 1.1 for paths.

Conjecture 1.2 (Kouider–Lonc, 1999). *Let ℓ be a positive integer. If G is a 2ℓ -regular graph, then G admits a balanced P_ℓ -decomposition.*

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One of the authors [2] proved the following weakening of Conjecture 1.2: for every positive integers ℓ and g such that $g \geq 3$, there exists an integer $m_0 = m_0(\ell, g)$ such that, if G is a $2m\ell$ -regular graph with $m \geq m_0$, then G admits a P_ℓ -decomposition \mathcal{D} such that every vertex of G is the end-vertex of exactly $2m$ paths of \mathcal{D} . In this paper we prove Conjecture 1.2 in the case $\ell = 4$.

1.1 Notation

A *trail* T is a graph for which there is a sequence $B = x_0 \cdots x_\ell$ of its vertices such that $E(T) = \{x_i x_{i+1} : 0 \leq i \leq \ell - 1\}$ and $x_i x_{i+1} \neq x_j x_{j+1}$, for every $i \neq j$. Such a sequence B of vertices is called a *tracking* of T and we say that T is the trail *induced* by the tracking B . We say that the vertices x_0 and x_ℓ are the *final vertices* of B . Given a tracking $B = x_0 \cdots x_\ell$ we denote by B^- the tracking $x_\ell \cdots x_0$. By abuse of notation, we denote by $V(B)$ and $E(B)$ the sets $\{x_0, \dots, x_\ell\}$ of vertices, and $\{x_i x_{i+1} : 0 \leq i \leq \ell - 1\}$ of edges of B , respectively. Moreover, we denote by \bar{B} the trail $(V(B), E(B))$, and by *length* of B we mean the length of \bar{B} . We also use ℓ -tracking to denote a tracking of length ℓ . A set of edge-disjoint trackings \mathcal{B} of a graph G is a *tracking decomposition* of G if $\cup_{B \in \mathcal{B}} E(B) = E(G)$. If every tracking of \mathcal{B} has length ℓ , we say that \mathcal{B} is an ℓ -tracking decomposition, and if every tracking of \mathcal{B} induces a path, we say that \mathcal{B} is a *path tracking decomposition*. For ease of notation, in this work we make no distinction between the trackings B and B^- in the following sense. Suppose $B \in \mathcal{B}$ is a tracking of a trail T ; when we need to choose a tracking of T we choose between B and B^- conveniently.

An *orientation* O of a subset E' of edges of G is an attribution of a direction (from one vertex to the other) to each edge of E' . If an edge xy is directed from x to y in O , we say that xy *leaves* x and *enters* y . Given a vertex v of G , we denote by $d_O^+(v)$ (resp. $d_O^-(v)$) the number of edges leaving (resp. entering) v with respect to O . We say that O is *Eulerian* if $d_O^+(v) = d_O^-(v)$ for every vertex v of G . We also denote by O^- , called *reverse orientation*, the orientation of E' such that if $xy \in E'$ is directed from x to y in O , then xy is directed from y to x in O^- .

Suppose that every tracking in \mathcal{B} has length at least 2. We consider an orientation O of a set of edges of G as follows. For each tracking $B = x_0 \cdots x_\ell$ in \mathcal{B} , we orient $x_0 x_1$ from x_1 to x_0 , and $x_{\ell-1} x_\ell$ from $x_{\ell-1}$ to x_ℓ . Given a vertex v of G , we denote by $\mathcal{B}(v)$ the number of edges of G directed towards v in O (i.e., $\mathcal{B}(v) = d_O^-(v)$) and by $\text{Hang}(v, \mathcal{B})$ the number of edges leaving v in O (i.e., $\text{Hang}(v, \mathcal{B}) = d_O^+(v)$). We say that an edge that leaves v in O is a *hanging* edge at v (this definition coincides with the definition of *pre-hanging* edge in [1]). We say that a tracking decomposition \mathcal{B} of G is *balanced* if $\mathcal{B}(u) = \mathcal{B}(v)$ for every $u, v \in V(G)$. It is clear that if \mathcal{B} is a balanced path tracking decomposition of G , then \mathcal{B} is a balanced path decomposition of G .

We say that a subgraph F of a graph G is a *factor* of G if $V(F) = V(G)$. If a factor F is r -regular, we say that F is an r -factor. Also, we say that a decomposition \mathcal{F} of G is an r -factorization if every element of \mathcal{F} is an r -factor.

1.2 Overview of the proof

Let G be an 8-regular graph. In Section 2 we use Petersen's 2-factorization theorem to obtain a 4-factorization $\{F_1, F_2\}$ of G . Then, we prove that F_1 admits a balanced P_2 -decomposition \mathcal{D} . Given an Eulerian orientation O to the edges of F_2 , we *extend* each path P of \mathcal{D} to a trail of length 4 using one outgoing edge of F_2 at each end-vertex of P (see Figure 1), thus obtaining a 4-tracking decomposition \mathcal{B} of G . We also prove that these extensions can be chosen such that no element of \mathcal{B} is a cycle of length 4. Lemma 2.7 shows that O can be chosen with some additional properties, which we call *good orientation* (see Definition 2.5), and Lemma 2.8 uses this special properties to show that the elements of \mathcal{B} that do not induce paths can be paired with paths of \mathcal{B} to form a new special element, which we call *exceptional extension* (see Figure 6). Thus, we can understand \mathcal{B} as a decomposition into paths and exceptional extensions. In Section 3, we show how to switch edges between the elements to obtain a decomposition into paths.

2 Decompositions into extensions

In this section we use Petersen's Factorization Theorem [11] to obtain a well-structured tracking decomposition of 8-regular graphs, called *exceptional decomposition into extensions*.

Theorem 2.1 (Petersen's 2-Factorization Theorem). *Every $2k$ -regular graph admits a 2-factorization.*

Let G be an 8-regular graph and let \mathcal{F} be a 2-factorization of G given by Theorem 2.1. By combining the elements of \mathcal{F} we obtain a decomposition of G into two 4-factors, say F_1 and F_2 . From now on, we fix such two 4-factors F_1 and F_2 . In the figures throughout the paper, we color the edges of F_1 with red, and the edges of F_2 with black. We first prove the following straightforward lemma.

Lemma 2.2. *If G is a 4-regular graph, then G admits a balanced P_2 -decomposition.*

Proof. Let G be a 4-regular graph and fix an Eulerian orientation O of its edges. For each vertex v of G , let P_v be the path consisting of the two edges of G that leave v in O . The set $\{P_v : v \in V(G)\}$ is a balanced P_2 -decomposition of G . \square

Now, let \mathcal{D}_1 be a balanced P_2 -decomposition of F_1 , O be an orientation of the edges of F_2 , and $B = x_0x_1x_2x_3x_4$ be a 4-tracking in G . We say that B is a (\mathcal{D}_1, O) -extension if $x_1x_2x_3 \in \mathcal{D}_1$, x_1x_0 is directed from x_1 to x_0 , and x_3x_4 is directed from x_3 to x_4 . We note that if T is a (\mathcal{D}_1, O) -extension, then exactly one of the following holds: (a) T is a path of length 4; (b) T contains a triangle; (c) T is a cycle of length 4 (see Figure 1). We say that a tracking decomposition \mathcal{B} of G is a *decomposition into (\mathcal{D}_1, O) -extensions* if every element of \mathcal{B} is a (\mathcal{D}_1, O) -extension. We omit \mathcal{D}_1 and O when it is clear from the context. The next result shows that every 8-regular graph admits a decomposition into extensions with no cycles. We denote by $\tau(\mathcal{B})$ the number of elements of \mathcal{B} that are cycles of length 4.

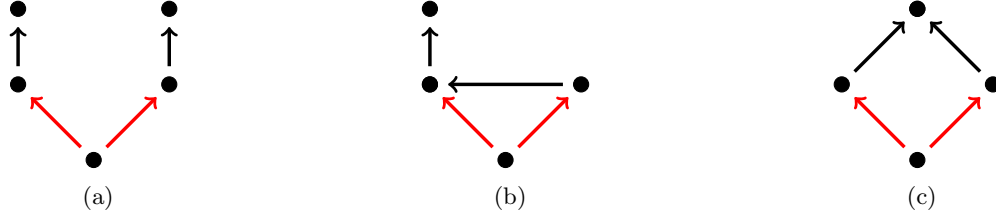


Figure 1: Extensions

Lemma 2.3. *Let G be an 8-regular graph, F be a 4-factor of G , \mathcal{D} be a balanced P_2 -decomposition of F , and O be an Eulerian orientation of the edges of $G - E(F)$. Then G admits a decomposition into (\mathcal{D}, O) -extensions with no cycles.*

Proof. Let G , \mathcal{D} , and O be as in the statement, and put $H = G - E(F)$. First, we prove that G admits a decomposition into (\mathcal{D}, O) -extensions. Indeed, since \mathcal{D} is balanced, $\mathcal{D}(v) = 2 = d_O^+(v)$ for every vertex v of G . Thus, we can extend every path $P = x_1x_2x_3$ in \mathcal{D} to a (\mathcal{D}, O) -extension $Q_P = x_0x_1x_2x_3x_4$ such that x_0x_1 and x_3x_4 are edges leaving x_1 and x_3 , respectively, and such that every edge of H is used exactly once. Therefore, $\{Q_P : P \in \mathcal{D}\}$ is a decomposition into (\mathcal{D}, O) -extensions.

Now, let \mathcal{B} be a decomposition of G into (\mathcal{D}, O) -extensions that minimizes $\tau(\mathcal{B})$. Suppose, for contradiction, that $\tau(\mathcal{B}) > 0$. Let $T = x_0x_1x_2x_3x_4$ be a cycle of length 4 in \mathcal{B} , where $x_1x_2x_3 \in \mathcal{D}$ and $x_0 = x_4$. Let $B = y_1y_2y_3$ be an element of \mathcal{D} such that $B \neq T$ and $y_1 = x_1$. Let $Q = y_0y_1y_2y_3y_4$ be the element of \mathcal{B} that contains B , and put $T' = y_0x_1x_2x_3x_4$ and $Q' = x_0y_1y_2y_3y_4$. Clearly, T' and Q' are (\mathcal{D}, O) -extensions, and T' is not a cycle. Moreover, if Q' is a cycle, then the edges x_0x_1 , x_3x_4 , and y_3y_4 are directed towards x_0 , which implies $d_O^-(x_0) \geq 3$, hence O is not an Eulerian orientation, a contradiction. Therefore, $\mathcal{B}' = \mathcal{B} - T + T' - Q + Q'$ is a decomposition into (\mathcal{D}, O) -extensions such that $\tau(\mathcal{B}') \leq \tau(\mathcal{B}) - 1$, a contradiction to the minimality of $\tau(\mathcal{B})$. \square

The following fact about decompositions into extensions are used in Section 3.

Fact 2.4. *Let G be an 8-regular graph, F be a 4-factor of G , \mathcal{D} be a balanced P_2 -decomposition of F , O be an Eulerian orientation of the edges of $G - E(F)$, and \mathcal{B} be a decomposition of G into (\mathcal{D}, O) -extensions. Then \mathcal{B} is balanced and $\text{Hang}(v, \mathcal{B}) = 2$ for every vertex v of G .*

Proof. Let G , \mathcal{D} , O , and \mathcal{B} be as in the statement, and put $H = G - E(F)$. Since O is an Eulerian orientation of H , $d_O^+(v) = d_O^-(v) = 2$ for every vertex v of G . By the definition of $\mathcal{B}(v)$, $\mathcal{B}(v) = d_O^-(v) = 2$ for every vertex v of G . Therefore, \mathcal{B} is balanced. By the definition of $\text{Hang}(v, \mathcal{B})$, $\text{Hang}(v, \mathcal{B}) = d_O^+(v) = 2$ for every vertex v of G . \square

2.1 Trapped subgraphs and good orientation

In this subsection we define two special concepts, namely, *trapped subgraphs* and *good orientations*, that are used throughout this section.

We say that an edge $uv \in F_2$ is *trapped* by \mathcal{D}_1 if there exists a path $P \in \mathcal{D}_1$ whose end-vertices are precisely u and v . Alternatively, we say that P *traps* the edge uv . Moreover, we say that

an induced path uvw in $G[F_2]$ is a \mathcal{D}_1 -*trapped* P_2 if the edges uv and vw are trapped by \mathcal{D}_1 and there exists a path in \mathcal{D}_1 whose end-vertices are precisely u and w (see Figure 2a); a triangle uvw in $G[F_2]$ is a \mathcal{D}_1 -*trapped triangle* (resp. \mathcal{D}_1 -*quasi-trapped triangle*) if all its edges (resp. two of its edges) are trapped by \mathcal{D}_1 (see Figure 2b and 2c); and a copy H of K_4 in $G[F_2]$ is a \mathcal{D}_1 -*trapped* K_4 if four of its edges are trapped by \mathcal{D}_1 (see Figure 2d). We omit the decomposition \mathcal{D}_1 when it is clear from the context. By a *trapped subgraph* of G we mean a subgraph of $G[F_2]$ that is a trapped P_2 , trapped triangle, or trapped K_4 . If a trapped edge e is not contained in any trapped subgraph or quasi-trapped triangle, then we say that e is a *free* trapped edge.

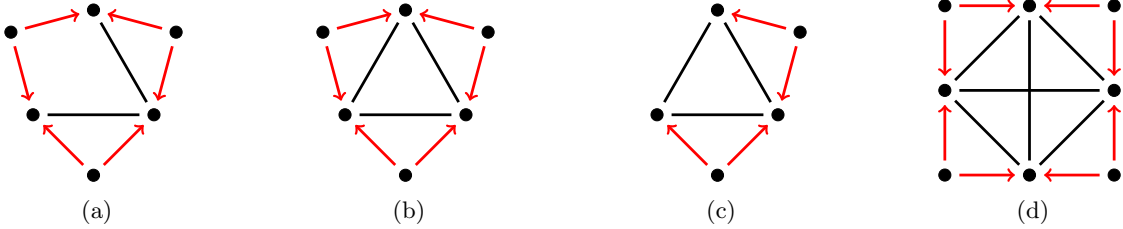


Figure 2: Trapped subgraphs of F_2

Let T be a trapped P_2 or quasi-trapped triangle of G , where uv and vw are the trapped edges of T . We say that an orientation O of the edges of T is *consistent* if $d_O^+(v) = d_O^-(v)$, otherwise, we say that O is *centered*. Now, we are able to define our special Eulerian orientation.

Definition 2.5. *Let G be an 8-regular graph, F be a 4-factor of G , \mathcal{D} be a balanced P_2 -decomposition of F . We say that an Eulerian orientation O of the edges of $G - E(F)$ is good if the following hold.*

- (i) *If T is a trapped P_2 of G , then O induces a consistent orientation of the edges of T ;*
- (ii) *if T is a trapped triangle of G , then O induces an Eulerian orientation of the edges of T ; and*
- (iii) *if T is a quasi-trapped triangle of G , then O induces an Eulerian orientation or a centered orientation of the edges of T (see Figure 3).*

Note that, since \mathcal{D}_1 is balanced, $\mathcal{D}_1(v) = 2$ for every $v \in V(G)$. Since each path P in \mathcal{D}_1 traps at most one edge of F_2 and $\mathcal{D}_1(v) = 2$ for every vertex v of G , each vertex v of G is incident to at most two trapped edges. Therefore, the subgraph F_2^t of F_2 induced by the \mathcal{D}_1 -trapped edges of F_2 is composed of vertex-disjoint paths and cycles. This implies the following fact. Two quasi-trapped triangles can have one

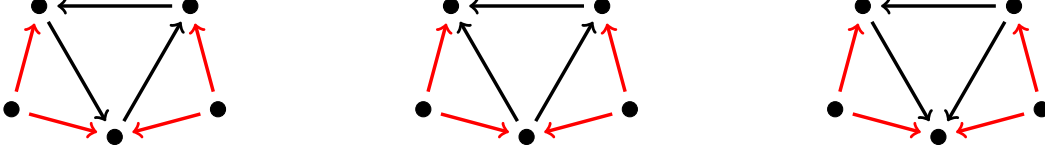


Figure 3: Eulerian and centered orientations of quasi-trapped triangles.

edge in common (see Figure 4), but each edge of F_2 is contained in at most one trapped subgraph of F_2 . Indeed, let T be a trapped subgraph of G , and T^t be the subgraph of F_2 induced by the trapped edges of T . Clearly, T^t is a subgraph of F_2^t . If T is a trapped triangle (resp. trapped K_4), then T^t is a triangle (resp. a cycle of length 4), hence T^t is a component of F_2^t . If T is a trapped P_2 , say $T = uvw$, then T^t is the path uvw , and $d_{F_2^t}(u) = d_{F_2^t}(w) = 1$. Therefore, T^t is a component of F_2^t . Since two components of F_2^t do not intercept, each edge of F_2 is contained in at most one trapped subgraph of G . Moreover, if T is a quasi-trapped triangle and intercepts a trapped subgraph T' , then T' must be a trapped K_4 .

In what follows we study sequences of quasi-trapped triangles. Note that two distinct quasi-trapped triangles have at most one trapped edge in common. We say that a sequence $S = T_1 \cdots T_k$ of quasi-trapped triangles is a *chain* of quasi-trapped triangles if T_i and T_{i+1} have a trapped edge in common for $i = 1, \dots, k-1$. If T_k and T_1 also have a trapped edge in common, then we say that S is a *closed* chain of quasi-trapped triangles, otherwise we say that S is *open*. The following fact about chains of quasi-trapped triangles are used in the proof of Lemma 2.7.

Fact 2.6. *Let $S = T_1 \cdots T_k$ be a chain of quasi-trapped triangles, and put $G_S = \cup_{i=1}^k T_i$. If S is open, then the trapped edges of G_S induce a Hamiltonian path $P = a_0 \cdots a_{k+1}$ in G_S , where a_1 and a_k are precisely the two vertices of odd degree in G_S . If S is closed then $k > 3$ and the following hold.*

- (i) *If $k = 4$, then G_S is a trapped K_4 ; and*
- (ii) *If $k > 4$, then G_S is a 4-regular graph and the trapped edges of G_S induce a Hamiltonian cycle in G_S*

We conclude that the subgraph F_2^t of F_2 induced by the trapped edges of F_2 can be decomposed into free trapped edges, trapped P_2 's, trapped triangles, trapped K_4 's, and maximal sequences of quasi-trapped triangles (that are not contained in trapped K_4 's).

Lemma 2.7. *Let G be an 8-regular graph, F be a 4-factor of G , \mathcal{D} be a balanced P_2 -decomposition of F , and put $H = G - E(F)$. Then, there is a good Eulerian orientation of the edges of H .*

Proof. Let G , \mathcal{D} , F , and H be as in the statement. In what follows, we construct a new Eulerian graph H^* from H , then we use an Eulerian orientation of H^* to obtain a good Eulerian orientation of H .

First, we deal with trapped subgraphs, and then with (chains of) quasi-trapped triangles. For every trapped P_2 , say $T = uvw$, where uv and vw are trapped edges, we split edges in the following way. We add a new vertex z_T , delete the edges uv and vw , and add the edges uz_T and z_Tw . For every trapped triangle or trapped K_4 , say T , delete all the trapped edges of T . It is clear that the graph H' obtained after these operations is Eulerian.

Now, let $S = T_1 \cdots T_k$ be a maximal chain of quasi-trapped triangles in H' , and put $G_S = \cup_{i=1}^k T_i$. If S is closed then $k > 4$, because H' has no trapped K_4 . By Fact 2.6, G_S is a 4-regular subgraph of H' and we delete the edges of G_S . Now, suppose that S is open. If $k = 1$, then delete the edges of T_1 . If $k > 1$, then by Fact 2.6, G_S contains a Hamiltonian path induced by the trapped edges in G_S , say $P = a_0 \cdots a_{k+1}$, where a_1 and a_k have odd degree in G_S . In this case, we delete the edges of G_S , and add the edge a_1a_k .

It is clear that the graph H^* obtained after these operations is again Eulerian. Therefore, let O^* be an Eulerian orientation of the edges of H^* . In what follows, we “undo” the operations above and obtain a good orientation O of the edges of H .

We must show how to orient each edge of H . If e is an edge in $E(H) \cap E(H^*)$ that is not contained in any trapped K_4 of G , then we direct e in O with the same direction e has in O^* . Let $S = T_1 \cdots T_k$ be a maximal chain of quasi-trapped triangles in H , and put $G_S = \cup_{i=1}^k T_i$. If $k = 1$, then G_S is a triangle, and

by the definition of H^* , no edge of G_S is in H^* , and we give an Eulerian orientation to the edges of G_S . If $k > 4$ and S is a closed chain, then S is not a trapped K_4 . By the definition of H^* , no edge of G_S is in H^* . By Fact 2.6, G_S is 4-regular and the trapped edges of G_S induce a Hamiltonian cycle $C_S = a_0 \cdots a_{k-1} a_0$ in G_S . Note that the edges in $G_S - E(C_S)$ are precisely $a_i a_{i+2}$ for $i = 0, \dots, k-1$, where $a_k = a_0$ and $a_{k+1} = a_1$. Thus, (in O) orient $a_i a_{i+1}$ from a_i to a_{i+1} , and $a_i a_{i+2}$ from a_{i+2} to a_i , for $i = 0, \dots, k-1$. Note that the triangle $a_i a_{i+1} a_{i+2}$ has an Eulerian orientation, for $i = 0, \dots, k-1$. Now, suppose that $k \geq 2$ and S is an open chain. By Fact 2.6, G_S contains a Hamiltonian path, say $P = a_0 \cdots a_{k+1}$, where a_1 and a_k have odd degree in G_S . From the construction of H^* , we have $a_1 a_k \in E(H^*)$. Let O_S be the orientation of G_S where the edge $a_i a_{i+1}$ is oriented from a_i to a_{i+1} , for $i = 0, \dots, k$ and $a_i a_{i+2}$ is oriented from a_{i+2} to a_i , for $i = 0, \dots, k-1$ (see Figure 4). Note that the triangle $a_i a_{i+1} a_{i+2}$ has an Eulerian orientation in O_S . If $a_1 a_k$ is oriented from a_k to a_1 in O^* , then orient in O the edges of G_S according to O_S . Otherwise $a_1 a_k$ is oriented from a_1 to a_k in O^* , and we orient in O the edges of G_S according to O_S^- . We have just chosen a



Figure 4: Eulerian orientation of quasi-trapped triangles.

direction to every edge in $E(H') \setminus E(H^*)$, except the (not trapped) edges in trapped K_4 's. Thus, if T is a quasi-trapped triangle in H not contained in a trapped K_4 , then O induces an Eulerian orientation of the edges of T .

Now, let K be a trapped K_4 , and let $x_i x_{i+1}$ be the trapped edges of K , for $i = 0, 1, 2, 3$, where $x_4 = x_0$. By construction, H^* contains the edges $x_0 x_2$ and $x_1 x_3$. Suppose, without loss of generality, that in O^* the edge $x_0 x_2$ is directed from x_0 to x_2 , and the edge $x_1 x_3$ is directed from x_1 to x_3 . We orient the edges of K in O in the following way. The edge $x_0 x_2$ is directed from x_0 to x_2 , and the edge $x_1 x_3$ is directed from x_3 to x_1 (i.e., with the direction opposite to the direction of $x_1 x_3$ in O^*). Moreover, orient the trapped edges of K such that $x_1 x_2 x_3$ and $x_1 x_0 x_3$ are two directed paths from x_1 to x_3 , i.e., $x_i x_{i+1}$ is directed from x_i to x_{i+1} for $i = 1, 2$, and directed from x_{i+1} to x_i , for $i = 0, 3$ (see Figure 5). Note that the orientations induced by O of the triangles $x_0 x_1 x_3$ and $x_1 x_2 x_3$ are Eulerian and that the orientations induced by O of the triangles $x_0 x_2 x_3$ and $x_0 x_1 x_2$ are centered (see Item (iii) of Definition 2.5). We have chosen a direction to every edge in $E(H) \cap E(H')$.

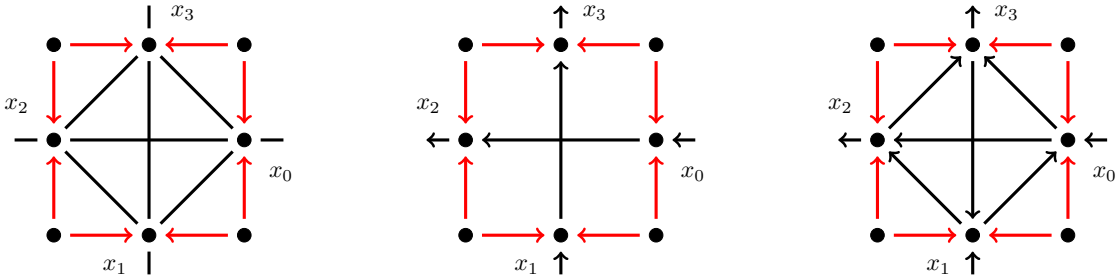


Figure 5: Good orientation of a trapped K_4 .

If T is a trapped triangle in H , then in O we orient the edges of T with any Eulerian orientation (see Item (ii) of Definition 2.5). Finally, let $T = uvw$ be a trapped P_2 in H . There exists a vertex z_T in H' incident exactly to the edges uz_T and $z_T w$. Thus $d_{O^*}^+(z_T) = d_{O^*}^-(z_T) = 1$. If uz_T is directed from u to z_T , we orient uv from u to v , and vw from v to w in O ; otherwise, we orient uv from v to u , and vw from w to v in O . This gives a consistent orientation to every trapped P_2 in H (see Item (i) of Definition 2.5). We conclude that O is a good Eulerian orientation of H . \square

2.2 Double-trapped edges and exceptional extensions

We say that an edge $uv \in F_2$ is *double-trapped* by \mathcal{D}_1 if there exist two distinct paths in \mathcal{D}_1 whose end-vertices are precisely u and v . Let $e \in F_2$ be double-trapped by \mathcal{D}_1 . If P_1 and P_2 are the paths of \mathcal{D}_1 that trap e , then $P_1 + e$ and $P_2 + e$ are triangles of G . Thus, if an edge $e \in F_2$ is double-trapped then it is the case for any orientation of F_2 . Therefore, if \mathcal{B} is a decomposition of G into (\mathcal{D}_1, O) -extensions for some Eulerian orientation O of F_2 , and T is the element of \mathcal{B} that contains e , then T contains a triangle.

Our next goal is to show that every 8-regular graph G admits a decomposition into paths of length 4 and a special object which we call *exceptional extension*.

Let $e \in F_2$ be double-trapped by the paths P_1 and P_2 of \mathcal{D}_1 , O be an Eulerian orientation of F_2 , and \mathcal{B} be a decomposition into (\mathcal{D}_1, O) -extensions of G with no cycles. Let T_i be the element of \mathcal{B} that contains P_i , for $i = 1, 2$. The *exceptional extension* that contains e is the pair $X = \{T_1, T_2\}$ (see Figure 6). It is clear that an exceptional extension contains a path of length 4 and a trail of length 4 that contains a triangle. We say that a decomposition into extensions \mathcal{B} with no cycles is *exceptional* if every element $T \in \mathcal{B}$ that contains a triangle is contained in exactly one exceptional extension.



Figure 6: Exceptional extensions

Now we are able to prove the main result of this section. Given a 4-tracking decomposition \mathcal{B} , we denote by $\tau'(\mathcal{B})$ the number of elements of \mathcal{B} that are paths.

Lemma 2.8. *Let G be an 8-regular graph, F be a 4-factor of G , \mathcal{D} be a balanced P_2 -decomposition of F , and O be a good Eulerian orientation of the edges of $G - E(F)$. Then G admits an exceptional decomposition into (\mathcal{D}, O) -extensions.*

Proof. Let G , F , \mathcal{D} , and P be as in the statement, By Lemma 2.3, there is a decomposition \mathcal{B} of G into (\mathcal{D}, O) -extensions with no cycles. Let \mathcal{B} be a decomposition of G into (\mathcal{D}, O) -extensions with no cycles that maximizes $\tau'(\mathcal{B})$. We claim that if there is an element T in \mathcal{B} that contains a triangle, then T contains a double-trapped edge.

Suppose, for a contradiction, that there is an element of \mathcal{B} , that contains a triangle. Let $T = x_0x_1x_2x_3x_4 \in \mathcal{B} \cup \mathcal{B}^-$, where $x_1x_2x_3 \in \mathcal{D}$, $x_4 = x_1$, and x_3x_4 is not double-trapped. Let $Q = y_0y_1y_2y_3y_4$ be an element of $\mathcal{B} \cup \mathcal{B}^-$ with $Q \neq T$, and such that $y_1y_2y_3 \in \mathcal{D}$ and $y_3 = x_3$, and put $T' = x_0x_1x_2x_3y_4$, $Q' = y_0y_1y_2y_3x_4$. Clearly, T' is a path or a cycle. Moreover, since x_3x_4 is not double-trapped, $y_1y_2y_3x_4$ is not a triangle, hence Q' contains a triangle only if Q contains the triangle $y_0y_1y_2y_3$. If T' and Q' are not cycles, then $\mathcal{B}' = \mathcal{B} - T + T' - Q + Q'$ is a decomposition into (\mathcal{D}, O) -extensions with no cycles, and $\tau'(\mathcal{B}') = \tau'(\mathcal{B}) + 1$, a contradiction to the maximality of $\tau'(\mathcal{B})$. In what follows we divide the proof on whether T' and Q' are cycles.

Case 1: T' is a cycle and Q' is not a cycle. In this case, we have $x_0 = y_4$ and $x_0x_1x_3y_4$ is a triangle in H . Thus, the edge x_0x_1 is not trapped, otherwise $x_0x_1x_3y_4$ is either a quasi-trapped triangle without Eulerian or centered orientation, or a trapped triangle without Eulerian orientation. Let $R = z_0z_1z_2z_3z_4$ an element in $\mathcal{B} \cup \mathcal{B}^-$ different from T , where $z_1z_2z_3 \in \mathcal{D}$ and $z_1 = x_1$. Since x_0x_1 is not trapped, we have $x_0 \neq z_3$. Moreover, we have $z_4 \neq x_0$, otherwise $d_O^-(x_0) \geq 3$. Therefore, $R' = x_0z_1z_2z_3z_4$ is a path. Also, since G has no parallel edges, we have $z_0 \neq x_3$ and $z_0 \neq y_4$. Thus, $T'' = z_0x_1x_2x_3y_4$ is a path. Therefore $\mathcal{B}' = \mathcal{B} - T + T'' - Q + Q' - R + R'$ is a decomposition into (\mathcal{D}, O) -extensions with no cycles, and $\tau'(\mathcal{B}') = \tau'(\mathcal{B}) + 1$, a contradiction to the maximality of $\tau'(\mathcal{B})$.

Case 2: Q' is a cycle and T' is not a cycle. Let $U = w_0w_1w_2w_3w_4$ be an element in $\mathcal{B} \cup \mathcal{B}^-$ different from Q , where $w_1w_2w_3 \in \mathcal{D}$ and $w_1 = y_1$. If $w_0 = y_3$, then $x_4x_3y_1y_0$ is either a quasi-trapped triangle without Eulerian or centered orientation, or a trapped triangle without Eulerian orientation. Moreover, $w_0 \neq x_4$ because G has no parallel edges. Therefore, $Q'' = w_0y_1y_2y_3x_4$ is a path. Now, let $U' = y_0w_1w_2w_3w_4$. If U' contains the triangle $y_0w_1w_2w_3$, then $w_1w_2w_3$ traps y_1y_0 and $y_1y_0x_3$ is either a trapped P_2 , without consistent orientation, or a trapped triangle without Eulerian orientation. Therefore, U' contains a triangle only if U contains the triangle $w_1w_2w_3w_4$. If U' is a cycle, then we have $w_4 = y_0$ and the edges x_3x_4 , y_1y_0 , and w_3w_4 are directed toward x_1 , hence $d_O^-(x_1) \geq 3$. Thus U' is not a cycle. Therefore $\mathcal{B}' = \mathcal{B} - T + T' - Q + Q'' - U + U'$ is a decomposition into (\mathcal{D}, O) -extensions with no cycles, and $\tau'(\mathcal{B}') = \tau'(\mathcal{B}) + 1$, a contradiction to the maximality of $\tau'(\mathcal{B})$.

Case 3: T' and Q' are cycles. In this case we have $x_4 \neq y_1$, otherwise y_0y_1 and x_0x_1 are parallel edges. Let $R = z_0z_1z_2z_3z_4$ and $U = w_0w_1w_2w_3w_4$ be elements in $\mathcal{B} \cup \mathcal{B}^-$ different from T and Q , where $z_1z_2z_3, w_1w_2w_3 \in \mathcal{D}$, and $z_1 = x_1$ and $w_1 = y_1$. We claim that $R \neq U$. Indeed, if $R = U$, then $z_1z_2z_3 = w_1w_2w_3$ and y_1y_0 is a trapped edge. Thus, $y_1y_0x_3$ is a trapped P_2 , without consistent orientation. Thus, analogously to cases 1 and 2, $R' = x_0z_1z_2z_3z_4$, $T'' = z_0x_1x_2x_3y_4$, $Q'' = w_0y_1y_2y_3x_4$ are paths, and $U' = y_0w_1w_2w_3w_4$ is not a cycle. Put $\mathcal{B}' = \mathcal{B} - T + T'' - Q + Q'' - R + R' - U + U'$. Again, \mathcal{B}' is a decomposition into (\mathcal{D}, O) -extensions with no cycles such that $\tau'(\mathcal{B}') \geq \tau'(\mathcal{B}) + 1$, a contradiction to the maximality of $\tau'(\mathcal{B})$.

We conclude that every element T of \mathcal{B} that contains a triangle contains a double-trapped edge, say e_T . Suppose that P_1 and P_2 are the elements of \mathcal{D} that trap e_T , where $P_1 \subset T$, and let T' be the element of \mathcal{B} that contains P_2 . The pair $X_T = \{T, T'\}$ is the exceptional extension that contains e_T . Thus, for every element T of \mathcal{B} that contains a triangle we obtain an exceptional extension X_T . It is clear that this exceptional extension is unique. Therefore, \mathcal{B} is exceptional. \square

3 Complete decompositions

In this section we relax the properties of the decomposition given by Lemma 2.8, and prove that every 8-regular graph admits a P_4 -decomposition. We start with the decomposition given by Lemma 2.8, and switch edges between the elements of this decomposition to obtain a decomposition containing only paths. Thus, the elements of the decompositions we consider here do not depend on \mathcal{D}_1 and O .

First we give some definitions. Let G be an 8-regular graph, and let \mathcal{B} be a 4-tracking decomposition of G with no cycles. An *exceptional pair* of \mathcal{B} is a pair of elements $X = \{T_1, T_2\}$ of $\mathcal{B} \cup \mathcal{B}^-$ such that $\bar{T}_1 \neq \bar{T}_2$, $T_1 = a_1b_1c_1d_1e_1$, $T_2 = a_2b_2c_2d_2e_2$, and $e_2 = b_2 = b_1$ and $d_1 = d_2$. We say that the vertices c_1 and c_2 are the *connection vertices* of X , and that d_2 is the *center* of X . By the definition of hanging edge, the edges a_1b_1 and a_2b_2 are hanging edges at b_1 , and d_1e_1 and d_2e_2 are hanging edges at d_1 . Note that an exceptional pair is exactly a pair of the underlying graphs of an exceptional extension.

The following definition presents the properties of the decompositions given by Lemma 2.8 that are used in the proof of our main result.

Definition 3.1. Let G be an 8-regular graph, and let \mathcal{B} be a balanced 4-tracking decomposition of G with no cycles. We say that \mathcal{B} is complete if the following hold.

- (i) $\text{Hang}(v, \mathcal{B}) \geq 1$, for every $v \in V(G)$;
- (ii) if $T \in \mathcal{B}$ contains a triangle, then T or T^- is contained in an exceptional pair of \mathcal{B} ; and
- (iii) if X is an exceptional pair of \mathcal{B} and P is an element of \mathcal{B} that contains the central vertex of X and a hanging edge at a connection vertex of X , then the central vertex of X is an end-vertex of P .

Now, we prove that the decomposition given by Lemma 2.8 induces a complete decomposition.

Lemma 3.2. If G is an 8-regular graph, then G admits a complete decomposition.

Proof. Let G be an 8-regular graph. By Petersen's Theorem [11], G admits a 2-factorization, say $\{F_1, F_2, F_3, F_4\}$. Thus, $F = F_1 + F_2$ and $H = F_3 + F_4$ are 4-factors of G . By Lemma 2.2, F admits a balanced P_2 -decomposition \mathcal{D} . By Lemma 2.7, there exists a good orientation O of the edges of H . By Lemma 2.8, G admits an exceptional decomposition \mathcal{B} into (\mathcal{D}, O) -extensions. In what follows, we prove that \mathcal{B} satisfies each item of Definition 3.1. By Fact 2.4, \mathcal{B} is balanced and $\text{Hang}(v, \mathcal{B}) = 2 \geq 1$ for every vertex v of G . This proves item (i) of Definition 3.1. Since \mathcal{B} is an exceptional decomposition \mathcal{B} has no cycles, and if $T \in \mathcal{B}$ contains a triangle, then T is contained in exactly one exceptional extension, which implies that T is contained in exactly one exceptional pair of \mathcal{B} . This proves item (ii) of Definition 3.1.

Now, suppose $X = \{T_1, T_2\}$ is an exceptional pair of \mathcal{B} , and $P \in \mathcal{B}$ is an element that contains a hanging edge xy at a connection vertex x of X . Suppose that P contains the center c of X . Note that there are two hanging edges at c contained in $E(T_1) \cup E(T_2)$. By the definition of X , we have $xc \in E(T_1) \cup E(T_2)$. Therefore, P contains a path $yxzc$, for some vertex z of G . If c is not an end-vertex of P , then there is another vertex z' such that P contains the path $yxzcz'$. Thus, P is exactly the tracking $yxzcz'$, and cz is a hanging edge of c . Therefore there are three hanging edges at c , a contradiction to $\text{Hang}(c, \mathcal{B}) = 2$. This proves item (iii) of Definition 3.1. \square

Suppose \mathcal{B} is a complete 4-tracking decomposition. Let $T = \{T_1, T_2\}$ be an exceptional pair, where $T_1 = a_1b_1c_1d_1e_1$ is a path. We say that the edge b_1d_1 is the *pivotal* edge of T_1 . Note that the pivotal edge of T_1 is an edge of T_2 . Therefore, T_1 is contained in at most one exceptional pair. Moreover, if w is the center of T , then $d_{T_1+T_2}(w) = 5$, hence w is not the center of any other exceptional pair.

Now we are able to prove our main theorem. Recall that $\tau'(\mathcal{B})$ is the number of trackings of \mathcal{B} that induce paths.

Theorem 3.3. *If G is an 8-regular graph, then G admits a balanced 4-tracking decomposition.*

Proof. Let G be an 8-regular graph. By Lemma 3.2, G admits a complete decomposition. Thus, let \mathcal{B} be a complete decomposition that maximizes $\tau'(\mathcal{B})$. We claim that \mathcal{B} is a path tracking decomposition. Suppose, for a contradiction, that \mathcal{B} contains an element T that contains a triangle. By item (ii) of Definition 3.1, T is contained in an exceptional pair of \mathcal{B} , say $X = \{T_1, T_2\}$ of \mathcal{B} . Suppose $T_1 = a_1b_1c_1d_1e_1$ and $T_2 = a_2b_2c_2d_2e_2$, where $e_2 = b_2 = b_1$ and $d_1 = d_2$. For ease of notation, let $b = b_1 = b_2$ and $d = d_1 = d_2$. By item (i) of Definition 3.1, $\text{Hang}(c_1, \mathcal{B}) \geq 1$ and $\text{Hang}(c_2, \mathcal{B}) \geq 1$. Thus, there is an element P_1 containing a hanging edge at c_1 , and an element P_2 containing a hanging edge at c_2 . We claim that at least one between P_1 and P_2 does not contain b . Indeed, suppose P_1 and P_2 contain b . By item (iii) of Definition 3.1, b is an end-vertex of P_1 and P_2 . But b is an end-vertex of T_2 . Therefore, $\mathcal{B}(b) \geq 3$, a contradiction to \mathcal{B} being a balanced decomposition.

Thus, we may assume that P_i does not contain b , and put $j = 3 - i$. Without loss of generality, let $P_i = a_3b_3c_3d_3e_3$, where $b_3 = c_i$ and a_3b_3 is a hanging edge at c_i (otherwise we have $P_i^- = a_3b_3c_3d_3e_3$, where $b_3 = c_i$ and a_3b_3 is a hanging edge at c_i). Put $P' = bb_3c_3d_3e_3$, and note that since P_i does not contain b , P' contains a triangle only if P_i contains the triangle $b_3c_3d_3e_3$. Now we show how to decompose the subgraph of G induced by $E(T_1) + E(T_2) - c_ib + c_ia_3$ into paths of length 4. We divide the proof into two cases, whether $a_2 = e_1$ or $a_2 \neq e_1$. If $a_2 = e_1$, then since \mathcal{B} is balanced, we have $a_2 \neq a_3$. Put $T'_1 = a_3c_idba_2$ and $T'_2 = a_1bc_jde_1$ (see Figure 7). Now, suppose $a_2 \neq e_1$. Since \mathcal{B} is balanced, we have $a_3 \neq a_1$ or $a_3 \neq a_2$. Say $a_3 \neq a_k$, where $k \in \{1, 2\}$, and put $l = 3 - k$. Note that, since T_1 is a path, we have $a_1 \neq e_1$, hence $a_k, a_l \neq e_1$. We put $T'_1 = a_3c_idba_k$ and $T'_2 = a_lbc_jde_1$.

It is clear that in the above two cases T'_1 and T'_2 are paths of length 4. Let $\mathcal{B}' = \mathcal{B} - P_i + P' - T_1 + T'_1 - T_2 + T'_2$ (note that we supposed that $P_1, T_1, T_2 \in \mathcal{B}$, otherwise, we use P_i^-, T_1^-, T_2^- , conveniently). We have $\tau'(\mathcal{B}') \geq \tau'(\mathcal{B}) + 1$. Now, we verify that \mathcal{B}' is a complete decomposition. Clearly, \mathcal{B}' is balanced, since $\mathcal{B}'(v) = 2$ for every vertex v of G .

(i) It is clear that the only edge that is a hanging edge in \mathcal{B} but not in \mathcal{B}' is db . But d_1e_1 is a hanging edge at d , thus $\text{Hang}(d, \mathcal{B}') \geq 1$. Moreover, if $v \neq d$, and e is a hanging edge at v in \mathcal{B} , then e is a hanging edge at v in \mathcal{B}' , hence $\text{Hang}(v, \mathcal{B}') \geq \text{Hang}(v, \mathcal{B}) \geq 1$ for every vertex $v \neq d$.

(ii) Let Q be an element of \mathcal{B}' that contains a triangle. If $Q \notin \{P', T'_1, T'_2\}$, then Q is an element of $\mathcal{B} \cup \mathcal{B}^-$, hence, by item (ii) of Definition 3.1, Q is contained in at least one exceptional pair of \mathcal{B} , say



Figure 7: Decomposing $T_1 + T_2 + P_i$ when $a_2 = e_1$.

$X_R = \{Q, W\}$ (otherwise Q^- is contained in an exceptional pair $X_R = \{Q^-, R\}$). As noted before, X_R is the only exceptional pair of \mathcal{B} containing W . Therefore, $W \neq T_1$. If $W = P_i$, then $\{Q, P'\}$ is an exceptional pair of \mathcal{B}' . Now, suppose $Q \in \{P', T'_1, T'_2\}$. Thus, $Q = P'$, because T'_1 and T'_2 do not contain triangles. As noted before, P' contains a triangle only if P_i contains the triangle $b_3c_3d_3e_3$. Note that $\{P_i, T_j\}$ is not an exceptional pair of \mathcal{B} . Thus, there is a path W in \mathcal{B} such that $X_R = \{P_i, W\}$ is an exceptional pair of \mathcal{B} . Therefore, $\{P', W\}$ is an exceptional pair of \mathcal{B}' .

(iii) Let $X' = \{Q, W\}$ be an exceptional pair of \mathcal{B}' with central vertex z , and let R be an element of \mathcal{B}' that contains a hanging edge at a connection vertex, say c , of X' and contains z . It is clear that $z \neq c$. Since \mathcal{B} is complete, if $Q, W, R \notin \{P', T'_1, T'_2\}$, then z is an end-vertex of R . Now, we claim that $Q, W \neq T'_1, T'_2$. Indeed, since the pivotal edge of T'_1 is in $E(T'_2)$, if $Q = T'_1$, then $W = T'_2$, but T'_1 and T'_2 are paths, hence $\{T'_1, T'_2\}$ is not an exceptional pair of \mathcal{B}' . Analogously, we have $Q \neq T'_2$. Therefore, P' is the only element in $\{P', T'_1, T'_2\}$ that is contained in an exceptional pair of \mathcal{B}' . In what follows we first study the cases where $R \in \{P', T'_1, T'_2\}$, and then the cases where $R \notin \{P', T'_1, T'_2\}$.

Suppose $R = P'$ and recall that $P' = bb_3c_3d_3e_3$. In this case, we have $Q, W \in \mathcal{B}$ and $c \in \{b_3, d_3\}$. Thus, P_i contains (in \mathcal{B}) a hanging edge at c . Since \mathcal{B} is complete, either $z \notin V(P_i)$ or z is an end-vertex of P_i . The only vertex in $V(P') - V(P_i)$ is b , which is the central vertex of $\{T_1, T_2\}$. Thus, if $z \notin V(P_i)$, then $z = b$, hence z is the central vertex of at least two exceptional pairs of \mathcal{B} , a contradiction. Therefore, we may assume that z is an end-vertex of P_i , i.e., $z \in \{a_3, e_3\}$. Suppose $z = a_3$. By the construction of P' , we have $a_3 \in V(P')$ if and only if P_i contains the triangle $a_3b_3c_3d_3$. In this case, $z = a_3 = d_3$, hence $c \neq d_3$ because $z \neq c$. Thus $c = b_3$, hence $b_3a_3 \in E(P_i)$ and $cz \in E(Q) \cup E(W)$ are parallel edges, a contradiction to G being simple. Thus, $z = e_3$ is an end-vertex of P' .

Now, we study the cases where $R = T'_1$ or $R = T'_2$. First, note that since $(E(Q) \cup E(W)) \cap (E(T_1) \cup E(T_2)) = \emptyset$, $d_{E(Q) \cup E(W)}(z) = d_{E(T_1) \cup E(T_2)}(b) = 5$, and $d_{E(T_1) \cup E(T_2)}(d) = 4$, we have $z \neq b, d$, otherwise $d(z) \geq 9$, a contradiction to G being 8-regular. Suppose that $R = T'_1$. In this case, we have $R = a_3c_idba_k$, where $k \in \{1, 2\}$, hence $c = c_i$ or $c = b$. Since $z \neq b, d$, if $c = c_i$, then $z \in \{a_3, a_k\}$. If $c = b$ and $z = c_i$, then $bc_i \in E(T'_2)$ and $cz \in E(Q) \cup E(W)$ are parallel edges, a contradiction. Thus, if $c = b$, then $z \in \{a_3, a_k\}$. In both cases we have $z \in \{a_3, a_k\}$, which are precisely the end-vertices of R . Suppose that $R = T'_2$. In this case, we have $R = a_lbc_jde_1$, where $l \in \{1, 2\}$. Thus, $c = b$ or $c = d$. In any of these cases, $z \neq c_j$, otherwise $cc_j \in E(T'_2)$ and $cz \in E(Q) \cup E(W)$ would be parallel edges, a contradiction. Since $z \neq b, d$, we have $z \in \{a_l, e_1\}$, which are precisely the end-vertices of R .

As noted before, at least one between Q, W, R must be in $\{P', T'_1, T'_2\}$. We already studied the cases there $R \in \{P', T'_1, T'_2\}$. Now it remains to prove the case where $R \notin \{P', T'_1, T'_2\}$. Also, we know that $T'_1, T'_2 \notin \{Q, W\}$. Thus $P' \in \{Q, W\}$ and $R \neq T'_1, T'_2$. Suppose, without of generality, that $Q = P'$. In this case, we have $W, R \notin \{P', T'_1, T'_2\}$. Since $\{P', W\}$ is an exceptional pair in \mathcal{B}' , $\{P_i, W\}$ is an exceptional pair in \mathcal{B} , because the middle edges, b_3c_3 and c_3d_3 , of P_i and P' are the same. Moreover, z is the central vertex of $\{P_i, W\}$ and c is a connection vertex of $\{P_i, W\}$. Thus, since \mathcal{B} is complete and R contains z , z is an end-vertex of R . Therefore, \mathcal{B}' satisfies item (iii) of Definition 3.1.

We conclude that \mathcal{B}' is a complete decomposition such that $\tau'(\mathcal{B}') > \tau'(\mathcal{B})$, a contradiction to the maximality of $\tau'(\mathcal{B})$. \square

Corollary 3.4. *If G is an 8-regular graph, then G admits a balanced P_4 -decomposition.*

4 Concluding remarks

In this paper we prove Conjecture 1.2 for paths of length 4. This result improves the previous result [10] that, for paths of length 4, states that triangle-free 8-regular graphs admit balanced P_4 -decompositions. We believe that the technique presented here can be modified to improve the girth condition for $\ell > 4$, or to prove Conjecture 1.1 for trees of diameter 4.

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